§6.1 cont'd

Alternate Method to find a determinant

given $A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 3 & 9 & 5 & 15 \\ 0 & 2 & 1 & 1 \\ 0 & 4 & 2 & 3 \end{bmatrix} \leftarrow$ 1ST row become coefficients

4x4

$\det A = 1 \det \begin{bmatrix} 9 & 5 & 15 \\ 2 & 1 & 1 \\ 4 & 2 & 3 \end{bmatrix} - 3 \det \begin{bmatrix} 3 & 5 & 15 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} + 1 \det \begin{bmatrix} 3 & 9 & 15 \\ 0 & 2 & 1 \\ 0 & 4 & 3 \end{bmatrix} - 4 \det \begin{bmatrix} 3 & 9 & 5 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$

alternate signs

$= 9 \det \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} - 5 \det \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} + 15 \det \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} - 3 \left(3 \det \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} - 5 \det \cancel{\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}} + 15 \det \cancel{\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}} \right) +$ 0

$3 \det \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} - 9 \cancel{\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}} + 15 \det \cancel{\begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}} - 4 \left(3 \det \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} - 0 + 0 \right)$

combine like terms

$= 0 - 2(6-4) + 3(4-4)$

$= -2(2) = \boxed{-4}$

given 5x5 matrix

$\begin{array}{ccccc} - & - & - & - & - \end{array}$

$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \\ 2 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix} \begin{matrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 7 & 6 & 5 & 4 \\ 2 & 1 & 2 & 3 \\ 5 & 6 & 7 & 8 \end{matrix}$

$\begin{array}{ccccc} - & - & - & - & - \end{array}$

UGH ...

can use Sarrus' rule →
re-write 5-1 = 4 cols
$\boxed{m-1}$

$\det A = 1(7)(5)(3)(9) + 2(8)(4)(4)(5) + \ldots$

same A then $A^T = \begin{bmatrix} 1 & 6 & 7 & 2 & 5 \\ 2 & 7 & 6 & 1 & 6 \\ 3 & 8 & 5 & 2 & 7 \\ 4 & 9 & 4 & 3 & 8 \\ 5 & 8 & 3 & 4 & 9 \end{bmatrix} \begin{matrix} 1 & 6 & 7 & 2 \\ 2 & 7 & 6 & 1 \\ 3 & 8 & 5 & 2 \\ 4 & 9 & 4 & 3 \\ 5 & 8 & 3 & 4 \end{matrix}$

$\det A^T = 1(7)(5)(3)(9) + \ldots$

thm: if A is a square matrix, then $\det(A^T) = \det A$

ex. find det A given $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

↑ not upper triangular ↑ is an upper triangular

↑ not upper triangular matrix

↑ is an upper triangular matrix

$\det A = \det A^T = 1(2)(3)(4) = \boxed{24}$

Determinant of $n \times n$ Matrix

2×2 : $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\text{re-label}} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

know $\det A = ad - bc$

$\det A = a_{11}a_{22} - a_{12}a_{21}$

$\oplus \begin{bmatrix} \bullet & \\ & \bullet \end{bmatrix}$ $\ominus \begin{bmatrix} & \bullet \\ \bullet & \end{bmatrix}$

2 choices
1 choice

$2 \cdot 1 = 2!$
↑
n

3×3 : $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$

$\det A = a_{11}a_{22}a_{33} + \ldots$

$\oplus \begin{bmatrix} \bullet & & \\ & \bullet & \\ & & \bullet \end{bmatrix}$ $\oplus \begin{bmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{bmatrix}$ $\oplus \begin{bmatrix} & & \bullet \\ \bullet & & \\ & \bullet & \end{bmatrix}$

each time
choose exactly one row/one column

$\ominus \begin{bmatrix} & & \bullet \\ & \bullet & \\ \bullet & & \end{bmatrix}$ $\ominus \begin{bmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{bmatrix}$ $\ominus \begin{bmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{bmatrix}$

each set of choices is a pattern, P

simplest pattern is diagonal pattern

First column: 3 choices
Second column: 2 choices
Third column: 1 choice

choices for all 3 columns:
$(3)(2)(1) = 3! = 6$
↑ Fundamental Counting Rule

recall: when rows are swapped, negate the determinant

let prod P := product of chosen entries within a pattern

ex. in a 3×3 matrix one sample prod $P = a_{11}a_{22}a_{33}$
" prod $P = a_{12}a_{23}a_{31}$

now, start to build a strategy for finding determinant of $n \times n$ matrix

$$\det A = \sum \pm \text{prod} P$$

↑ let's get a better understanding of alternating signs:

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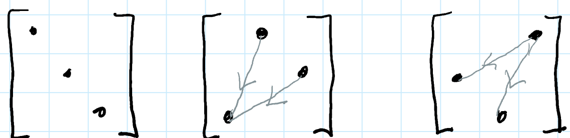
ex. $\det \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array}\begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix}$ swap II, III ← would prefer this to be upper triangular matrix

$= - \det \begin{bmatrix} 0 & a_{12} & 0 \\ a_{31} & 0 & 0 \\ 0 & 0 & a_{23} \end{bmatrix}$ swap $= + \det \begin{bmatrix} a_{31} & 0 & 0 \\ 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \end{bmatrix} = \boxed{a_{31} a_{12} a_{23}}$

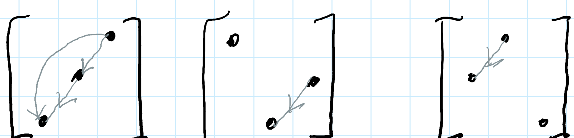
upper triangular :)

2 swaps or 2 inversions

shortcut: invert if one entries is to the right and above another



no inversions   2 inversions   2 inversions   (+) even # of inversions



3 inversions   1 inversion   1 inversion   (−) odd # of inversions

fine tune formula: $\det A = \sum \underbrace{(-1)^{\# \text{ inversions}}}_{\operatorname{sgn} P} \operatorname{prod} P$ pattern

sgn P signature of P

$$\boxed{\begin{array}{l} \det A = \sum \operatorname{sgn} P \operatorname{prod} P \\ \text{where sum is taken over all } n! \text{ patterns, } P, \text{ in } n \times n \text{ matrix } A \end{array}}$$

ex. find det A given

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$ ← only 1 pattern P

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7 inversions

$\det A = (-1)^7\, 3(2)(1)(8)(5)(2)$

$= \boxed{-480}$

ex. find det A given $\begin{bmatrix} 6 & 0 & 1 & 0 & 0 \end{bmatrix}$

why can't we use 1 ?

$\begin{bmatrix} 6 & 0 & 1 & 0 & 0 \end{bmatrix}$

ex. find det A given

$$A = \begin{bmatrix} 6 & 0 & 1 & 0 & 0 \\ 9 & 3 & 2 & 3 & 7 \\ 8 & 0 & 3 & 2 & 9 \\ 0 & 0 & 4 & 0 & 0 \\ 5 & 0 & 5 & 0 & 1 \end{bmatrix}$$

max of 2 potential patterns

1 inversion

why can't we use 1 ?

$$\begin{bmatrix} 6 & 0 & 1 & 0 & 0 \\ 9 & 3 & 2 & 3 & 7 \\ 8 & 0 & 3 & 2 & 9 \\ 0 & 0 & \cancel{4} & 0 & 0 \\ 5 & 0 & 5 & 0 & 1 \end{bmatrix}$$

can't use...

$$\det A = -6(3)(2)(4)(1) = \boxed{-144}$$